ACADEMIA ROMÂNĂ
Revue Roumaine de Chimie
http://web.icf.ro/rrch/

Rev. Roum. Chim.,
2020, *65*(11), 1045-1055
DOI: 10.33224/rrch.2020.65.11.09

(*a, b*)-ZAGREB INDEX OF SOME SPECIAL GRAPHS

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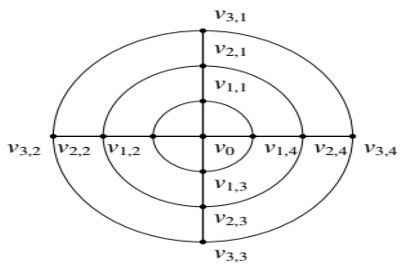
Received October 19, 2020

In the past few years, graph theory has emerged as one of the most powerful mathematical tools to model many types of relations and process dynamics in computer science, biological and social systems. Generally, a graph is depicted as a set of nodes which is called vertices connected by lines are called edges. A topological index is the numerical parameter of a graph that characterizes its topology and it is usually graph invariant. In this paper, we compute some important classes vertex degree-based graph invariants using the (*a, b*)-Zagreb index of some special graphs such as the co-normal product of graphs, concentric wheels graph and intersection graph.



$W_{4,3}$

INTRODUCTION

Recently, there are various graph operations or derived graphs plays an important role to model the geometrical structure of any communication system including internet which is based on graph. The logical set up of a computer is designed with the help of graph. Therefore, it is not surprising that different graph operations have been used to many diverse problems in computer science and the field of chemical sciences to model chemical compounds. A graph $\Gamma \cong (V, E)$ consists of a set denoted by V, and a collection $E \subseteq V \times V$, of unordered pairs $\{\alpha, \beta\}$ of distinct elements from V. The set V is called vertex set and each element of V is called vertex or nodes and the set E is called edge set and each element of it is called edge or link. In this paper, we consider all graphs are finite, simple and connected. The degree of a vertex $\alpha \in \Gamma$ is the number of edges in Γ associated with it and is denoted as $d_\Gamma(\alpha)$ or simply $d(\alpha)$. A topological index of a graph is numeric quantity obtained from that graph by mathematically which is fixed under graph isomorphism.

Topological indices are used for example in the development of quantitative structure-activity relationships (QSAR) in which the biological activity or other properties of molecules are correlated with their chemical structures. In this paper, we compute the (a, b) −Zagreb index of co-normal product of graphs,[1] wheels graph and intersection graph[2-3] and derived some other degree based graph invariants as an applications of (a, b)-Zagreb index for some particular values of a and b. The Zagreb indices were introduced in,[4] to compute the total π −electron energy (ϵ) of carbon atoms in 1972 and are defined as

$$M_1(\Gamma) = \sum_{\alpha \in V(\Gamma)} d(\alpha)^2 = \sum_{\alpha\beta \in E(\Gamma)} \big(d(\alpha) + d(\beta)\big)$$

and

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$$M_2(\Gamma) = \sum_{\alpha\beta \in E(\Gamma)} d(\alpha)d(\beta).$$

We refer to,[5-9] for more details about these indices. In,[10] Furtula and Gutman reinvestigate the "forgotten topological index" in 2015 and is defined as

$$F(\Gamma) = \sum_{\alpha \in V(\Gamma)} d(\alpha)^3 = \sum_{\alpha\beta \in E(\Gamma)} (d(\alpha)^2 + d(\beta)^2).$$

For more details about this index we refer our reader to.[11-13] The redefined Zagreb index was introduced in 2013, by Ranjini *et al.* in[14] and this index is defined as

$$ReZM(\Gamma) = \sum_{\alpha\beta \in E(\Gamma)} (d(\alpha)d(\beta))(d(\alpha) + d(\beta)).$$

We refer to,[15-17] for further results about this index. Followed by first Zagreb index and F-index Li and Zheng was first introduced the general Zagreb index in[18] and is defined as follows

$$M^a(\Gamma) = \sum_{\alpha \in V(\Gamma)} d(\alpha)^a$$

where, $a \neq 0,1$ and $a \in \mathbb{R}$. In details about this index we refer to.[19-22] General form of Randicc' index is defined as

$$R_a = \sum_{\alpha\beta \in E(\Gamma)} \left(d(\alpha)d(\beta)\right)^a$$

where, $a \neq 0$ and $a \in \mathbb{R}$ and it is introduced by Gutman and Lepovi c' [23] in 2001. Interested reader to see,[24-25] for details about this index. The symmetric division deg index is defined as

$$SDD(\Gamma) = \sum_{\alpha\beta \in E(\Gamma)} \left(\frac{d(\alpha)}{d(\beta)} + \frac{d(\beta)}{d(\alpha)}\right).$$

We encourage our reader to,[26-27] for some interesting results about this index. Azari *et al.* in 2011, introduced the $(a,b)-$Zagreb index followed by Zagreb indices in,[28] which is defined as

$$Z_{a,b}(\Gamma) = \sum_{\alpha\beta \in E(\Gamma)} (d(\alpha)^a d(\beta)^b + d(\alpha)^b d(\beta)^a).$$

Some our recent study about this index, we encourage our reader to.[29-31]

MAIN RESULTS

In the following sub sections, we compute the (a,b)-Zagreb index of co-normal product of graphs, wheels graph and intersection graph. With the help of this (a,b)-Zagreb index we also calculate some more graph invariants in the form of corollaries for some particular values of a and b. First we consider the co-normal product of graphs.

Co-normal product of graphs

Let us suppose that Γ_1 and Γ_2 be two graphs with vertex sets $V(\Gamma_1) = \{v_1, v_2, v_3, \cdots, v_m\}$ and $V(\Gamma_2) = \{u_1, u_2, u_3, \cdots, u_n\}$ respectively. The co-normal product of Γ_1 and Γ_2 is denoted by $\Gamma_{1[\Gamma_2]}$ with the vertex set $V(\Gamma_1) \times V(\Gamma_2) = \{v_{hk} = (v_h, u_k): v_h \in V(\Gamma_1),$ and two vertices v_{hk} and v_{rs} of $\Gamma_{1[\Gamma_2]}$ are connected if v_h is connected to v_r in Γ_1 or v_k is connected to v_s in Γ_2. In this paper, we compute (a,b)-Zagreb index of co-normal product between two path graphs P_m and P_x where, m and x denotes the number of vertices in P_m and P_x respectively. The total number of vertices in $P_{m[P_x]}$ is mx and the total number of edges in $P_{m[P_x]}$ is $\{m^2(x-1) + x^2(m-1) - 2(m-1)(x-1)\}$ or $(m^2 l_2 + x^2 l_1 - 2l_1 l_2)$ where, l_1 and l_2 are the number of edges in P_m and P_x respectively. Thus $|V(P_{m[P_x]})| = mx$ and $|E(P_{m[P_x]})| = m^2(x-1) + x^2(m-1) - 2(m-1)(x-1)$. The edge set of $P_{m[P_x]}$ based on degree of the end vertices can be partitioned into 9-distinct subsets which are shown in the following Table 2. An example of $P_{5[P_4]}$ is shown in Figure 1. Let us consider $P_{m[P_x]} = G_1$ (Say).

Table 1

In the following table we derived various degree based graph invariants from the (a,b)-Zagreb index for some particular values of *a* and *b*

Graph invariants	Corresponding (a,b)-Zagreb index
$M_1(\Gamma)$	$Z_{1,0}(\Gamma)$
$M_2(\Gamma)$	$\frac{1}{2}Z_{1,1}(\Gamma)$

Table 1 (continued)

$F(\Gamma)$	$Z_{2,0}(\Gamma)$
$ReZM(\Gamma)$	$Z_{2,1}(\Gamma)$
$M^{a}(\Gamma)$	$Z_{a-1,0}(\Gamma)$
$R_{a}(\Gamma)$	$\frac{1}{2}Z_{a,a}(\Gamma)$
$SDD(\Gamma)$	$Z_{1,-1}(\Gamma)$

Table 2

Edge partition of $P_{m[P_x]}$ $(m, x \geq 3)$

$(d(\alpha), d(\beta)), \alpha\beta \in E(P_{m[P_x]})$	**Total number of edges**
$d(\alpha) = (m + x - 1)$, $d(\beta) = 2 + 2(x - 1) + (m - 2)$	8
$d(\alpha) = (m + x - 1)$, $d(\beta) = 2 + 2(m - 1) + (x - 2)$	8
$d(\alpha) = (m + x - 1)$, $d(\beta) = 2m + 2(x - 2)$	$4(m - 2 + x - 3)$
$d(\alpha) = 2 + 2(x - 1) + (m - 2) = d(\beta)$	$4m + 12$
$d(\alpha) = 2 + 2(m - 1) + (x - 2) = d(\beta)$	$4x + 12$
$d(\alpha) = 2 + 2(x - 1) + (m - 2)$ $d(\beta) = 2 + 2(m - 1) + (x - 2)$	$4(m + x - 5)$
$d(\alpha) = 2 + 2(x - 1) + (m - 2)$ $d(\beta) = 2m + 2(x - 2)$	$4(m - 3)(x - 3) + 2(m - 2)^2$
$d(\alpha) = 2 + 2(m - 1) + (x - 2)$ $d(\beta) = 2m + 2(x - 2)$	$4(m - 3)(x - 3) + 2(x - 2)^2$
$d(\alpha) = 2m + 2(x - 2) = d(\beta)$	$(m - 2)^2(x - 3)$ $+ (m - 3)\{2(x - 3) + (x - 4)^2\}$

Theorem 1. The (a, b) −Zagreb index $Z_{a,b}(G_1) = D\ (Say)$ is given by

$$D = 8\big((m+x-1)^{a} \cdot (2x+m-2)^{b} + (m+x-1)^{b} \cdot (2x+m-2)^{a}\big) + 8\big((m+x-1)^{a} \cdot (2m+x-2)^{b} + (m+x-1)^{b} \cdot (2m+x-2)^{a}\big) + (4m+4x-20)\big((m+x-1)^{a} \cdot (2m+2x-4)^{b} + (m+x-1)^{b} \cdot (2m+2x-4)^{a}\big) + (8m-24)(2x+m-2)^{(a+b)} + (8x-24)(x+2m-2)^{(a+b)} + (4m+4x-20)\big((2x+m-2)^{a} \cdot (x+2m-2)^{b} + (2x+m-2)^{b} \cdot (x+2m-2)^{a}\big) + \{2(m-2)^2 + 4(x-3)(m-3)\}\big((2x+m-2)^{a} \cdot (2x+2m-4)^{b} + (2x+m-2)^{b} \cdot (2x+2m-4)^{a}\big) + \{2(x-2)^2 + 4(x-3)(m-3)\}\big((x+2m-2)^{a} \cdot (2x+2m-4)^{b} + (x+2m-2)^{b} \cdot (2x+2m-4)^{a}\big) + 2[(m-2)^2(x-3)\{2(x-3) + (x-4)^2\}(m-3)](2x+2m-4)^{(a+b)} \tag{1}$$

Proof. From definition of general Zagreb index we get,

$$D = \sum_{\alpha\beta \in E(G_1)} \left(d(\alpha)^{a} d(\beta)^{b} + d(\alpha)^{b} d(\beta)^{a}\right)$$

$$= \sum_{\alpha\beta \in E_1(G_1)} \big((m+x-1)^{a} \cdot (2x+m-2)^{b} + (m+x-1)^{b} \cdot (2x+m-2)^{a}\big)$$

$$+\sum_{\alpha\beta\in E_2(G_1)}\left((m+x-1)^a\cdot(2m+x-2)^b+(m+x-1)^b\cdot(2m+x-2)^a\right)$$

$$+\sum_{\alpha\beta\in E_3(G_1)}\left((m+x-1)^a\cdot(2m+2x-4)^b+(m+x-1)^b\cdot(2m+2x-4)^a\right)$$

$$+\sum_{\alpha\beta\in E_4(G_1)}\left((2x+m-2)^a\cdot(2x+m-2)^b+(2x+m-2)^b\cdot(2x+m-2)^a\right)$$

$$+\sum_{\alpha\beta\in E_5(G_1)}\left((x+2m-2)^a\cdot(x+2m-2)^b+(x+2m-2)^b\cdot(x+2m-2)^a\right)$$

$$+|E_7(G_1)|\left((2x+m-2)^a\cdot(2m+2x-4)^b+(2x+m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+|E_8(G_1)|\left((x+2m-2)^a\cdot(2m+2x-4)^b+(x+2m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+|E_9(G_1)|\left((2m+2x-4)^a\cdot(2m+2x-4)^b+(2m+2x-4)^b\cdot(2m+2x-4)^a\right)$$

$$=8\left((m+x-1)^a\cdot(2x+m-2)^b+(m+x-1)^b\cdot(2x+m-2)^a\right)$$

$$+8\left((m+x-1)^a\cdot(2m+x-2)^b+(m+x-1)^b\cdot(2m+x-2)^a\right)$$

$$+(4m+4x-20)\left((m+x-1)^a\cdot(2m+2x-4)^b+(m+x-1)^b\cdot(2m+2x-4)^a\right)$$

$$+(4m-12)\left((2x+m-2)^a\cdot(2x+m-2)^b+(2x+m-2)^b\cdot(2x+m-2)^a\right)$$

$$+(4x-12)\left((x+2m-2)^a\cdot(x+2m-2)^b+(x+2m-2)^b\cdot(x+2m-2)^a\right)$$

$$+(4m+4x-20)\left((2x+m-2)^a\cdot(x+2m-2)^b+(2x+m-2)^b\cdot(x+2m-2)^a\right)$$

$$+\left(2(m-2)^2+4(x-3)(m-3)\right)\left((2x+m-2)^a\cdot(2m+2x-4)^b+(2x+m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+\left(2(x-2)^2+4(x-3)(m-3)\right)\left((x+2m-2)^a\cdot(2m+2x-4)^b+(x+2m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+\left((m-2)^2(x-3)+(m-3)\{2(x-3)+(x-4)^2\}\right)\left((2m+2x-4)^a\cdot(2m+2x-4)^b+(2m+2x-4)^b\cdot(2m+2x-4)^a\right)$$

$$+\sum_{\alpha\beta\in E_6(G_1)}\left((2x+m-2)^a\cdot(x+2m-2)^b+(2x+m-2)^b\cdot(x+2m-2)^a\right)$$

$$+\sum_{\alpha\beta\in E_7(G_1)}\left((2x+m-2)^a\cdot(2m+2x-4)^b+(2x+m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+\sum_{\alpha\beta\in E_8(G_1)}\left((x+2m-2)^a\cdot(2m+2x-4)^b+(x+2m-2)^b\cdot(2m+2x-4)^a\right)$$

$$+\sum_{\alpha\beta\in E_9(G_1)}\left((2m+2x-4)^a\cdot(2m+2x-4)^b+(2m+2x-4)^b\cdot(2m+2x-4)^a\right)$$

$$=|E_1(G_1)|\left((m+x-1)^a\cdot(2x+m-2)^b+(m+x-1)^b\cdot(2x+m-2)^a\right)$$

$$+|E_2(G_1)|\left((m+x-1)^a\cdot(2m+x-2)^b+(m+x-1)^b\cdot(2m+x-2)^a\right)$$

$$+|E_3(G_1)|\left((m+x-1)^a\cdot(2m+2x-4)^b+(m+x-1)^b\cdot(2m+2x-4)^a\right)$$

$$+|E_4(G_1)|\left((2x+m-2)^a\cdot(2x+m-2)^b+(2x+m-2)^b\cdot(2x+m-2)^a\right)$$

$$+|E_5(G_1)|\left((x+2m-2)^a\cdot(x+2m-2)^b+(x+2m-2)^b\cdot(x+2m-2)^a\right)$$

$$+|E_6(G_1)|\left((2x+m-2)^a\cdot(x+2m-2)^b+(2x+m-2)^b\cdot(x+2m-2)^a\right)$$
$$+|E_7(G_1)|\left((2x+m-2)^a\cdot(2m+2x-4)^b+(2x+m-2)^b\cdot(2m+2x-4)^a\right)$$
$$+|E_8(G_1)|\left((x+2m-2)^a\cdot(2m+2x-4)^b+(x+2m-2)^b\cdot(2m+2x-4)^a\right)$$
$$+|E_9(G_1)|\left((2m+2x-4)^a\cdot(2m+2x-4)^b+(2m+2x-4)^b\cdot(2m+2x-4)^a\right)$$
$$=8\left((m+x-1)^a\cdot(2x+m-2)^b+(m+x-1)^b\cdot(2x+m-2)^a\right)$$
$$+8\left((m+x-1)^a\cdot(2m+x-2)^b+(m+x-1)^b\cdot(2m+x-2)^a\right)$$
$$+(4m+4x-20)\left((m+x-1)^a\cdot(2m+2x-4)^b+(m+x-1)^b\cdot(2m+2x-4)^a\right)$$
$$+(4m-12)\left((2x+m-2)^a\cdot(2x+m-2)^b+(2x+m-2)^b\cdot(2x+m-2)^a\right)$$
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$$+(4m+4x-20)\left((2x+m-2)^a\cdot(x+2m-2)^b+(2x+m-2)^b\cdot(x+2m-2)^a\right)$$
$$+\left(2(m-2)^2+4(x-3)(m-3)\right)\left((2x+m-2)^a\cdot(2m+2x-4)^b+(2x+m-2)^b\cdot(2m+2x-4)^a\right)$$
$$+\left(2(x-2)^2+4(x-3)(m-3)\right)\left((x+2m-2)^a\cdot(2m+2x-4)^b+(x+2m-2)^b\cdot(2m+2x-4)^a\right)$$
$$+\left((m-2)^2(x-3)+(m-3)\{2(x-3)+(x-4)^2\}\right)\left((2m+2x-4)^a\cdot(2m+2x-4)^b+\right.$$
$$\left.(2m+2x-4)^b\cdot(2m+2x-4)^a\right)$$

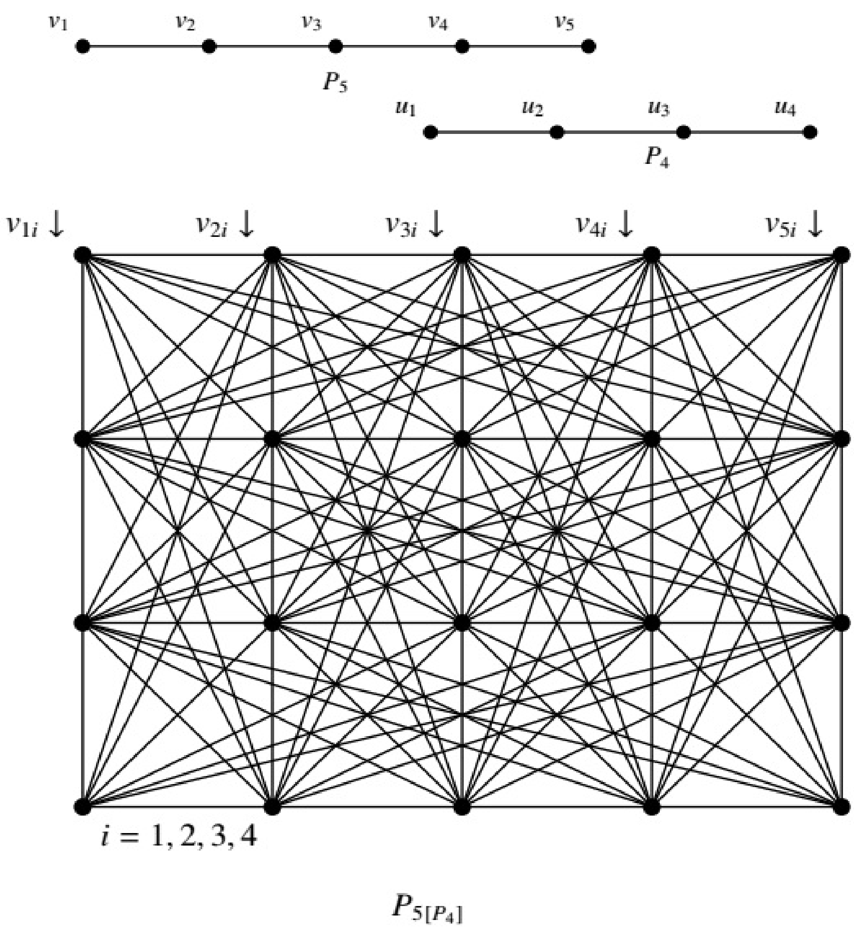


Fig. 1 – The example of co-normal product between P_5 and P_4.

Hence, the theorem.

Corollary 1. From equation 1, we derived the following results,

$$M_1(G_1) = Z_{1,0}(G_1) = 8(3x + 2m - 3) + 8(2x + 3m - 3) + (4m + 4x - 20)(3m + 3x - 5) + (8m - 24)(2x + m - 2) + (8x - 24)(x + 2m - 2) + (4m + 4x - 20)(3m + 3x - 4) + \left(2(m-2)^2 + 4(x-3)(m-3)\right)(4x + 3m - 6) + \left(2(x-2)^2 + 4(x-3)(m-3)\right)(3x + 4m - 6) + ((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\})(4m + 4x - 8),$$

$$\begin{aligned} M_2(G_1) = \frac{1}{2} Z_{1,1}(G_1) &= 8(m + x - 1)(2x + m - 2) + 8(m + x - 1)(2m + x - 2) \\ &+ (4m - 12)(2x + m - 2)^2 + (4m + 4x - 20)(m + x - 1)(2m + 2x - 4) \\ &+ (4x - 12)(x + 2m - 2)^2 + (4m + 4x - 20)(2x + m - 2)(2m + x - 2) \\ &+ \left(2(m-2)^2 + 4(m-3)(x-3)\right)(2x + m - 2)(2m + 2x - 4) \\ &+ \left(2(x-2)^2 + 4(m-3)(x-3)\right)(x + 2m - 2)(2m + 2x - 4) \\ &+ ((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\})(2m + 2x - 4)^2, \end{aligned}$$

$$F(G_1) = Z_{2,0}(G_1) = 8\{(m + x - 1)^2 + (2x + m - 2)^2\} + 8\{(m + x - 1)^2 + (2m + x - 2)^2\} + (4m + 4x - 20)((m + x - 1)^2 + (2m + 2x - 4)^2) + (0m - 24)(2x + m - 2)^2 + (8x - 24)(x + 2m - 2)^2 + (4m + 4x - 20)((2x + m - 2)^2 + (x + 2m - 2)^2) + \left(2(m-2)^2 + 4(m-3)(x-3)\right)((2x + m - 2)^2 + (2m + 2x - 4)^2) + (2(x-2)^2 + 4(m-3)(x-3))((x + 2m - 2)^2 + (2m + 2x - 4)^2) + 2((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\})(2m + 2x - 4)^2,$$

$$ReZM(G_1) = Z_{2,1}(G_1) = 8(m + x - 1)(2x + m - 2)(3x + 2m - 3) + 8(m + x - 1)(2m + x - 2)(2x + 3m - 3) + (4m + 4x - 20)(m + x - 1)(2m + 2x - 4)(3m + 3x - 5) + (8m - 24)(2x + m - 2)^3 + (8x - 24)(x + 2m - 2)^3 + (4m + 4x - 20)(2x + m - 2)(x + 2m - 2)(3m + 3x - 4) + \left(2(m-2)^2 + 4(x-3)(m-3)\right)(2x + m - 2)(2m + 2x - 4)(4x + 3m - 6) + \left(2(x-2)^2 + 4(x-3)(m-3)\right)(x + 2m - 2)(2m + 2x - 4)(3x + 4m - 6) + 2((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\})(2m + 2x - 4)^3,$$

$$M^a(G_1) = Z_{a-1,0}(G_1) = 8((m + x - 1)^{a-1} + (2x + m - 2)^{a-1}) + 8((m + x - 1)^{a-1} + (2m + x - 2)^{a-1}) + (4m + 4x - 20)((m + x - 1)^{a-1} + (2m + 2x - 4)^{a-1}) + (8m - 24)(2x + m - 2)^{a-1} + (8x - 24)(x + 2m - 2)^{a-1} + (4m + 4x - 20)((2x + m - 2)^{a-1} + (x + 2m - 2)^{a-1}) + \left(2(m-2)^2 + 4(m-3)(x-3)\right)((2x + m - 2)^{a-1} + (2m + 2x - 4)^{a-1}) + \left(2(x-2)^2 + 4(m-3)(x-3)\right)((2m + x - 2)^{a-1} + (2m + 2x - 4)^{a-1}) + 2((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\})(2m + 2x - 4)^{a-1},$$

$$R_a(G_1) = \frac{1}{2} Z_{a,a}(G_1) = 8(m + x - 1)^a(2x + m - 2)^a + 8(m + x - 1)^a(2m + x - 2)^a + (4m - 12)(2x + m - 2)^{2a} + (4m + 4x - 20)(m + x - 1)^a(2m + 2x - 4)^a + (4x - 12)(x + 2m - 2)^{2a} + (4m + 4x - 20)(2x + m - 2)^a(2m + x - 2)^a + \left(2(m-2)^2 + 4(m-3)(x-3)\right)(2x + m - 2)^a(2m + 2x - 4)^a + \left(2(x-2)^2 + 4(m-3)(x-3)\right)(x + 2m - 2)^a(2m + 2x - 4)^a + ((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\}) + (2m + 2x - 4)^{2a},$$

$$SDD(G_1) = Z_{1,-1}(G_1) = 8((m + x - 1)^1(2x + m - 2)^{-1} + (m + x - 1)^{-1}(2x + m - 2)^1) + 8((m + x - 1)^1(2m + x - 2)^{-1} + (m + x - 1)^{-1}(2m + x - 2)^1) + (4m + 4x - 20)((m + x - 1)^1(2m + 2x - 4)^{-1} + (m + x - 1)^{-1}(2m + 2x - 4)^1) + (8m - 24) + (8x - 24) + (4m + 4x - 20)((2x + m - 2)^1(x + 2m - 2)^{-1} + (2x + m - 2)^1(x + 2m - 2)^{-1}) + \left(2(m-2)^2 + 4(m-3)(x-3)\right)((2x + m - 2)^1(2m + 2x - 4)^{-1} + (2x + m - 2)^{-1}(2m + 2x - 4)^1) + \left(2(x-2)^2 + 4(m-3)(x-3)\right)((x + 2m - 2)^1(2m + 2x - 4)^{-1} + (x + 2m - 2)^{-1}(2m + 2x - 4)^1) + 2((m-2)^2(x-3) + (m-3)\{2(x-3) + (x-4)^2\}).$$

Table 3

Edge partition of $W_{m,n}$ $(n \geq 3)$

$(d(\alpha), d(\beta))$: $\alpha\beta \in E(G_2)$	Total number of edges
$d(v_0) = m, d(\beta) = 4$	m
$d(\alpha) = 4 = d(\beta)$	$2mn - 3m$
$d(\alpha) = 3 = d(\beta)$	m
$d(\alpha) = 4, d(\beta) = 3$	m

Concentric wheels graph

Let us consider n concentric cycles with v_0 is the center and let each cycles divided into m parts that is each cycles contains m vertices. We denote r^{th} vertex of k^{th} cycle with $v_{r,k}$, for $1 \leq r \leq m, 1 \leq k \leq n$. In concentric wheels graph r^{th} vertex of k^{th} cycle is connected to corresponding r^{th} vertex of $(k-1)^{th}$ cycle for $1 \leq r \leq m, 2 \leq k \leq n$ and v_0 is connected to each r^{th} vertex of nearest cycle that is for $k = 1$. This graph is denoted as $W_{m,n}$, the total number of vertices in $W_{m,n}$ is $mn + 1$ and the total number of edges in $W_{m,n}$ is $2mn$. The edge partition of $W_{m,n}$ is shown in Table 3. The example of a concentric wheels graph for $n = 3$ and $m = 4$ is shown in Figure 2. Let us consider $W_{m,n} = G_2$ (Say).

Theorem 2. The (a,b) −Zagreb index of concentric wheels graph G_2 is given by

$$Z_{a,b}(G_2) = \left(m^{a+1} \cdot 4^b + m^{b+1} \cdot 4^a\right) + (2mn - 3m).2^{(2(a+b)+1)} + 2m \cdot 3^{a+b} + m(4^a \cdot 3^b + 4^b \cdot 3^a). \quad (2)$$

Proof. From the Table 3 and **by** definition of (a,b) −Zagreb index, we get

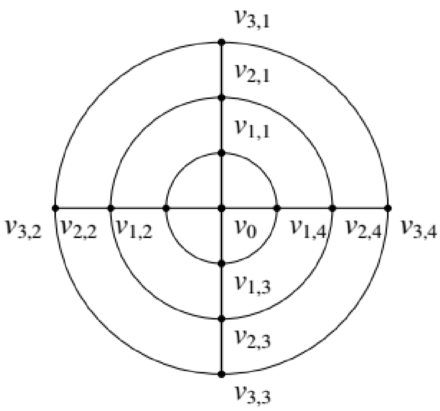
$$Z_{a,b}(G_2) = \sum_{\alpha\beta \in E(G_2)} \left(d(\alpha)^a d(\beta)^b + d(\alpha)^b d(\beta)^a\right)$$

$$= \sum_{\alpha\beta \in E_1(G_2)} \left(m^a \cdot 4^b + m^b \cdot 4^a\right) + \sum_{\alpha\beta \in E_2(G_2)} \left(4^a \cdot 4^b + 4^b \cdot 4^a\right)$$

$$+ \sum_{\alpha\beta \in E_3(G_2)} \left(3^a \cdot 3^b + 3^b \cdot 3^a\right) + \sum_{\alpha\beta \in E_4(G_2)} \left(4^a \cdot 3^b + 4^b \cdot 3^a\right)$$

$$= |E_1(G_2)|\left(m^a \cdot 4^b + m^b \cdot 4^a\right) + |E_2(G_2)|\left(4^a \cdot 4^b + 4^b \cdot 4^a\right)$$

$$+ |E_3(G_2)|\left(3^a \cdot 3^b + 3^b \cdot 3^a\right) + |E_4(G_2)|\left(4^a \cdot 3^b + 4^b \cdot 3^a\right)$$



$W_{4,3}$

Fig. 2 – The example of a concentric wheels graph $W_{m,n}$ $(m = 4, n = 3)$.

$$= m\left(m^a \cdot 4^b + m^b \cdot 4^a\right) + (2mn - 3m)\left(4^a \cdot 4^b + 4^b \cdot 4^a\right)$$
$$+ m\left(3^a \cdot 3^b + 3^b \cdot 3^a\right) + m\left(4^a \cdot 3^b + 4^b \cdot 3^a\right)$$

Hence, flows the results as shown in equation 2.

Corollary 2. In the following corollary, we derived some degree based topological indices for some particular values of a and b by using equation 2,

(i) $M_1(G_2) = Z_{1,0}(G_2) = \mathrm{m}(\mathrm{m} + 16\mathrm{n} - 7)$,

(ii) $M_2(G_2) = \frac{1}{2} Z_{1,1}(G_2) = 4\mathrm{m}^2 + 32\mathrm{mn} - 27\mathrm{m}$,

(iii) $F(G_2) = Z_{2,0}(G_2) = \mathrm{m}^3 + 64\mathrm{mn} - 37\mathrm{m}$,

(iv) $ReZM(G_2) = Z_{2,1}(G_2) = 4\mathrm{m}^3 + 16\mathrm{m}^2 + 25\mathrm{mn} - 246\mathrm{m}$,

(v) $M^a(G_2) = Z_{a-1,0}(G_2) = \left(\mathrm{m}^a + \mathrm{m} \cdot 4^{(a-1)}\right) + (2\mathrm{mn} - 3\mathrm{m}) \cdot 2^{(2a-1)} + 2\mathrm{m}.3^{(a-1)} + \mathrm{m}\left(4^{(a-1)} + 3^{(a-1)}\right)$,

(vi) $R_a(G_2) = \frac{1}{2} Z_{a,a}(G_2) = 4^a.\mathrm{m}^{(a+1)} + (2\mathrm{mn} - 3\mathrm{m}) \cdot 2^{4a} + \mathrm{m} \cdot 3^a(3^a + 4^a)$,

(vii) $SDD(G_2) = Z_{1,-1}(G_2) = \frac{1}{4}(\mathrm{m}^2 + 16) + 4\mathrm{mn} + \frac{\mathrm{m}}{12}$.

Intersection graph

Let A be a non empty finite set and $H = \{A_1, A_2, \cdots, A_n\}$ be a family of non empty distinct subsets of A such that $A = \bigcup_{k=1}^{n} A_k$. The intersection graph of H is denoted as $\Omega(H)$ such that $V(\Omega(H))$, with two vertices A_h and A_k in $\Omega(H)$ are connected whenever $A_h \cap A_k \neq \phi$ for $h \neq k$. Thus a graph Γ is an intersection graph on A if there exist a family H of non empty distinct subsets of A for which $\Gamma \cong \Omega(H)$. In this paper, we consider a non empty finite set A of cardinality m and let H be the collections of all non empty distinct subsets of A of cardinality $p(1 < p < m)$ and is denoted by H_p. Let the intersection graph $\Omega(H_p)$ be denoted as Γ_p. The total number of vertices in Γ_p is $\binom{\mathrm{m}}{\mathrm{p}}$ that is $|V(\Gamma_p)| = \binom{\mathrm{m}}{\mathrm{p}}$ and

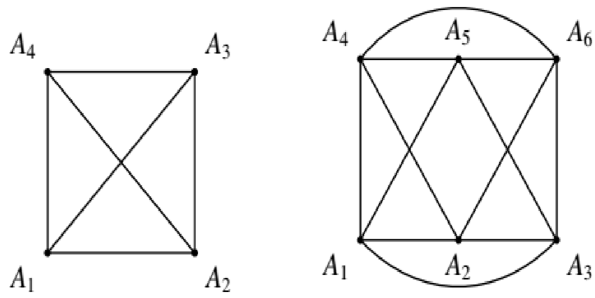
$$|E(\Gamma_p)| = \begin{cases} \frac{1}{2}\binom{\mathrm{m}}{\mathrm{p}}\left(\binom{\mathrm{m}}{\mathrm{p}} - \binom{\mathrm{m}-\mathrm{p}}{\mathrm{p}} - 1\right), & if\ m \geq 2p \\ \frac{1}{2}\binom{\mathrm{m}}{\mathrm{p}}\left(\binom{\mathrm{m}}{\mathrm{p}} - 1\right), & if\ m < 2p. \end{cases}$$

Lemma 1. The degree of each vertex $\alpha \in \Gamma_p$ is shown in the following lemma as follows:

$$d_{\Gamma_p}(\alpha) = \begin{cases} \left(\binom{\mathrm{m}}{\mathrm{p}} - \binom{\mathrm{m}-\mathrm{p}}{\mathrm{p}} - 1\right), & if\ m \geq 2p \\ \left(\binom{\mathrm{m}}{\mathrm{p}} - 1\right), & if\ m < 2p. \end{cases}$$

The examples of intersection graphs are shown in Figure 3, where figure (a) shows the example of $\Omega(H_3)$ or Γ_3 and figure (b) shows the example of $\Omega(H_2)$ or Γ_2. Here we consider $A = \{1,2,3,4\}$ and H_3, H_2 are respectively defined as

$$H_3 = \{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\},$$
$$H_2 = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}.$$



(*a*) (*b*)

Fig. 3 – The examples of intersection graphs $\Omega(H_3)$ and $\Omega(H_2)$.

Theorem 3. The (a,b) −Zagreb index of intersection graph Γ_p is given by

$$Z_{a,b}(\Gamma_p) = 2\left|E(\Gamma_p)\right|\left(d_{\Gamma_p}(\alpha)^{a+b}\right). \qquad (3)$$

Proof. From definition of (a,b) −Zagreb index and using Lemma 1, we get

$$Z_{a,b}(\Gamma_p) = \sum_{\alpha\beta\in E(\Gamma_p)} \left(d(\alpha)^a d(\beta)^b + d(\alpha)^b d(\beta)^a\right)$$

$$= \sum_{\alpha\beta\in E(\Gamma_p)} \left(2\cdot d(\alpha)^{a+b}\right)\ [as\ d(\alpha) = d(\beta)]$$

$$= 2\left|E(\Gamma_p)\right|\left(d(\alpha)^{a+b}\right).$$

Hence the theorem.

Corollary 3. From equation 3, we get the following corollary as shown in the following,

$$Z_{a,b}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^{a+b}, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)\left(\binom{m}{p} - 1\right)^{a+b}, & if\ m < 2p. \end{cases} \qquad (4)$$

Corollary 4. For some particular values of a and b we compute some other degree based topological indices from equation 4, in the following corollary as follows:

$$(i)\ M_1(\Gamma_p) = Z_{1,0}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^2, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^2, & if\ m < 2p. \end{cases}$$

$$(ii)\ M_2(\Gamma_p) = \frac{1}{2}Z_{1,1}(\Gamma_p) = \frac{1}{2}\begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^3, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^3, & if\ m < 2p. \end{cases}$$

$$(iii)\ F(\Gamma_p) = Z_{2,0}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^3, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^3, & if\ m < 2p. \end{cases}$$

$$(iv)\ ReZM(\Gamma_p) = Z_{2,1}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^4, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^4, & if\ m < 2p. \end{cases}$$

$$(v)\ M^a(\Gamma_p) = Z_{a-1,0}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^a, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^a, & if\ m < 2p. \end{cases}$$

$$(vi)\ R_a(\Gamma_p) = \frac{1}{2} Z_{a,a}(\Gamma_p) = \frac{1}{2}\begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right)^{(2a+1)}, & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right)^{(2a+1)}, & if\ m < 2p. \end{cases}$$

$$(vii)\ SDD(\Gamma_p) = Z_{1,-1}(\Gamma_p) = \begin{cases} \binom{m}{p}\left(\binom{m}{p} - \binom{m-p}{p} - 1\right), & if\ m \geq 2p \\ \binom{m}{p}\left(\binom{m}{p} - 1\right), & if\ m < 2p. \end{cases}$$

CONCLUSIONS

In this paper, we consider three types of special graphs such as co-normal product of graphs, concentric wheels graph and intersection graph. Based on case study we have classified the edge partitions of these graph operations and compute (*a*,*b*)-Zagreb index for these graphs. We also compute some other degree based graph invariants such as $M_1(\Gamma), M_2(\Gamma), F(\Gamma), ReZM(\Gamma), M^a(\Gamma), R_a(\Gamma)$ and $SDD(\Gamma)$ for some particular values of a and b. In future study, we will consider some other graph structures to compute this (a,b) –Zagreb index.

Acknowledgements. We would like to take this opportunity to acknowledge the time and effort devoted by reviewers to improve the manuscript. The first author would like to express special thanks of gratitude to the Council of Scientific and Industrial Research (CSIR), Government of India for their financial support under the grant no. 09/973(0016)/2017-EMR-I.

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